Tutorial 6 : Selected problems of Assignment 6

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Notation Throughout the tutorial, (X,d) is a (not necessarily complete) metric space with metric d. $E, F \subseteq X$ are nonempty subsets.

Q1) (Ex 6, Q4) (a) Show that $\overline{E} \cup \overline{F} = \overline{E \cup F}$ holds.

(b) Show that $\overline{E} \cap \overline{F} = \overline{E \cap F}$ does NOT hold in general.

Sol) (a) [$\subseteq$] Note that $\begin{cases} E \subseteq E \cup F \\ F \subseteq E \cup F \end{cases} \rightsquigarrow \begin{cases} \overline{E} \subseteq \overline{E \cup F} \\ \overline{F} \subseteq \overline{E \cup F} \end{cases} \rightsquigarrow \overline{E} \cup \overline{F} \subseteq \overline{E \cup F}$

[$\supseteq$] $\overline{E}, \overline{F}$ are closed $\rightsquigarrow$ $\overline{E} \cup \overline{F}$ is closed.

Therefore, $E \cup F \subseteq \overline{E} \cup \overline{F} \rightsquigarrow \overline{E \cup F} \subseteq \overline{E} \cup \overline{F}$.

(b) Consider $(X, d) = (\mathbb{R}, |\cdot|)$; $E = (0,1)$; $F = (1,2)$.

Then $LHS = \overline{E} \cap \overline{F} = [0,1] \cap [1,2] = \{1\}$

$RHS = \overline{E \cap F} = \overline{(0,1) \cap (1,2)} = \overline{\phi} = \phi$

$\therefore$ LHS $\neq$ RHS.

Q2) (Ex. 6, Q5) Recall the distance function to E, $\rho_E : (X,d) \to \mathbb{R}$

defined as $\rho_E(x) = d(x,E) := \inf_{z\in E}\{d(x,z)\}$. (See Lecture note § 2.2)

Show that $\overline{E} = \{x \in X \mid d(x,E) = 0\}$.

Sol) Let $A = \{x\in X \mid d(x,E) = 0\}$. Showing A is closed:

For any sequence $(x_n) \subseteq A$ such that $\lim_{n\to\infty} x_n = x$, by continuity of ρ_E,

$d(x,E) = \lim_n d(x_n,E) = 0$ (since $x_n \in E$). $\therefore x \in A$, hence A is closed

[$\subseteq$] Note that $E \subseteq A$. As A is closed, $\overline{E} \subseteq A$.

[$\supseteq$] Given $x \in A$, for any $n \in \mathbb{N}$, $\inf_{z\in E}\{d(x,z)\} < \frac{1}{n}$ $\rightsquigarrow$ there exists $z_n \in E$

such that $d(x,z_n) < \frac{1}{n}$. $\therefore \lim_{n\to\infty} d(x,z_n) = 0$. Hence $\lim_{n\to\infty} z_n = x$, i.e. $x \in \overline{E}$.

Q3) (Ex 6, Q6, 7) Assume $\mathring{E} \neq \phi$. (a) Show that $\mathring{E} = X \setminus \overline{(X \setminus E)}$

(b) Hence, show that $\mathring{E}$ is the largest open set contained in E, i.e.

(i) $\mathring{E}$ is an open subset of E.

(ii) For any open subset $G \subseteq E$, then $G \subseteq \mathring{E}$.

Sol) (a) [$\subseteq$] Given $z \in \mathring{E}$. Hence there exists $\rho > 0$ such that $B_\rho(z) \subseteq E$.

Suppose on the contrary $z \in \overline{X \setminus E}$, then there exists a sequence $(x_n) \subseteq X \setminus E$

such that $\lim_{n\to\infty} x_n = z$. Hence there exists $N \in \mathbb{N}$ such that $d(x_N, z) < \rho$

$\therefore x_N \in B_\rho(z) \subseteq E$. This is a contradiction. $\therefore z \in X \setminus \overline{(X \setminus E)}$.

[$\supseteq$] Given $y \in X \setminus \overline{(X \setminus E)}$, suppose on the contrary $y \notin \mathring{E}$.

Then for any $n \in \mathbb{N}$, $B_{\frac{1}{n}}(y) \nsubseteq E$. Hence there exists $z_n \in B_{\frac{1}{n}}(y)$

such that $z_n \notin E$. $\therefore \lim_{n\to\infty} z_n = y$, hence $y \in \overline{X \setminus E}$, contradiction

(b) (i) Since $\overline{X \setminus E}$ is closed, $\mathring{E} = X \setminus (\overline{X \setminus E})$ is open.

(ii) Given open $G \subseteq E \rightsquigarrow X \setminus G \supseteq X \setminus E \rightsquigarrow \overline{X \setminus G} \supseteq \overline{X \setminus E}$

$\rightsquigarrow G = X \setminus (X \setminus G) = \underset{\text{(since } X\setminus G \text{ is closed)}}{X \setminus (\overline{X \setminus G})} \subseteq X \setminus (\overline{X \setminus E}) = \mathring{E} \rightsquigarrow G \subseteq \mathring{E}$.